Composition Operators on Sobolev Spaces (with Applications to Spectral Theory and Nonlinear Elasticity).

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In the present talk we give a short review of the theory of composition operators on Sobolev spaces, its applications to spectral theory of elliptic operators and we will discuss a new point of view to nonlinear elasticity problems.

For any conformal homeomorphism $\frac{|\varphi_z'(x,y)|^2}{|J(x,y;\varphi)|} = 1$
Hence for any weakly differentiable function f we have

$$\|\nabla(f \circ \varphi) : L_2(\Omega)\|^2 = \int_\Omega |\nabla(f \circ \varphi)|^2 dvol$$

$$= \int_\Omega |\nabla(f \circ \varphi)|^2 \frac{|\varphi_z'(x,y)|^2}{|J(x,y;\varphi)|} |J(x,y;\varphi)| dvol$$

$$= \int_{\Omega'} |\nabla f|^2 dvol = \left\|\nabla f : L_2(\Omega')\right\|^2 .$$

It means that conformal mappings induces isometries of L_2 norms of gradients.

Recall that $Q := \frac{|\varphi'_z(x,y)|^2}{|J(x,y;\varphi)|}$ is a coefficient of quasiconformality (a dilatation), that equal 1 for conformal homeomorphisms and is uniformly bounded and bigger than 1 for quasiconformal homeomorphisms.
Its generalization $Q_p = \frac{|\varphi'_z(x,y)|^p}{|J(x,y;\varphi)|}$ for $p \neq 2$ is called a p-dilatation and corresponding homeomorhisms are called p-quasiconformal.

For dimension $n \geq 2$, $Q = \frac{|\varphi'_z(x,y)|^n}{|J(x,y;\varphi)|}$ is a coefficient of quasiconformality (dilatation).
The quantity $Q_p = \frac{|\varphi'_z(x,y)|^p}{|J(x,y;\varphi)|}$, $p \neq n$ is called a p-dilatation and corresponding homeomorhisms are called p-quasiconformal.

We study composition operators $\varphi^* : L^{1,p}(\Omega') \to L^{1,p}(\Omega)$ of uniform Sobolev spaces $L^{1,p}$ defined in space domains $\Omega, \Omega' \subset \mathbb{R}^n$ under an additional assumption that $\varphi : \Omega \to \Omega'$ are homeomorphisms.

Recall that the uniform Sobolev space $L^{1,p}(\Omega)$, $1 \leq p < \infty$, is defined as a seminormed space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|f \mid L^1_p(\Omega)\| = \left(\int_{\Omega} |\nabla f(x)|^p \, dvol \right)^{\frac{1}{p}}.$$

In the case $p = n$ the composition operators can be induced by quasiconformal homeomorphisms only (Vodop'janov, G. 1975). For $p \neq n$ the composition operators can be induced by p-quasiconformal homeomorphisms only (G.,Gurov, Romanov 1995). The class of p-quasiconformal homeomorphisms coincide with the class of quasiconformal homeomorphisms for $p = n$. For $p = n$ composition operators φ^* are invertible. For other p it is not correct.

For applications more useful to use composition operators $\varphi^* : L^{1,p}(\Omega') \to L^{1,q}(\Omega)$, $q \leq p$ that induced by so-called (p,q)-quasiconformal homeomorphism only. In the case $n-1 \leq q \leq p$ inverse homeomorphisms are $(\frac{q}{q-(n-1)}, \frac{p}{p-(n-1)})$-quasiconformal, that induce the composition operator $\varphi^* : L^{1,\frac{q}{q-(n-1)}}(\Omega) \to L^{1,\frac{p}{p-(n-1}}(\Omega')$. This is a new type of "duality". In all these cases descriptions of composition operators are exact (necessary and sufficient conditions).

Main applications are based on the composition operators theory for different p, q (Ukhlov, 1993, Ukhlov and Vodop'yanov, 2002) and the following diagramm introduced in (G. and Gurov, 1994, G. and Ukhlov, 2009):

$$\begin{array}{ccc} L^{1,p}(\Omega) & \overset{(\varphi^{-1})^*}{\longrightarrow} & L^{1,q}(\mathbb{D}) \\ \downarrow & & \downarrow \\ L^s(\Omega) & \overset{\varphi^*}{\longleftarrow} & L^r(\mathbb{D}) \end{array}$$

Here the operator φ^* defined by the composition rule $\varphi^*(f) = f \circ \varphi$ is a bounded composition operator on Lebesgue spaces induced by a homeomorphism φ from Ω to $\mathbb{D}$ and the bounded omposition operator $(\varphi^{-1})^*$ is defined by the composition rule $(\varphi^{-1})^*(f) = f \circ \varphi^{-1}$.

Recall the analytic (geometric) description of homeomorphisms that generate bounded composition operators (Ukhlov, 1993):

Composition Theorem. *A homeomorphism $\varphi:\Omega\to\Omega'$ between two domains $\Omega,\Omega'\subset\mathbb{R}^n$, $n\geq 2$, induces a bounded composition operator*

$$\varphi^*: L^1_p(\Omega')\to L^1_q(\Omega),\ \ 1\leq q<p<\infty,$$

if and only if $\varphi\in L^1_{1,loc}(\Omega)$, has finite distortion, and

$$K_{p,q}(f;\Omega)=\left(\int\limits_{\Omega}\left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|}\right)^{\frac{q}{p-q}}\,dx\right)^{\frac{p-q}{pq}}<\infty.$$

A mapping has finite distortion if at almost every point or Jacobian is positive or the derivative is equal to zero.

Brennan's conjecture is that for a conformal mapping $\varphi : \Omega \to \mathbb{D}$

$$\int_{\Omega} |\varphi'(x,y)|^{\delta}\, dxdy < +\infty, \quad \text{for all} \quad \frac{4}{3} < \delta < 4.$$

For the inverse conformal mapping $\psi = \varphi^{-1} : \mathbb{D} \to \Omega$ Brennan's conjecture states

$$\iint_{\mathbb{D}} |\psi'(u,v)|^{\gamma}\, dudv < +\infty, \quad \text{for all} \quad -2 < \gamma < \frac{2}{3}.$$

For bounded domains the conjecture is different $-2 < \gamma \leq 2$ and the right hand part $\gamma \leq 2$ is proved (G. Ukhlov 2012)

Equivalence Theorem. (G., Ukhlov 2012) *Brennan's Conjecture holds for a number $s \in (4/3; 4)$ if and only if any conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$ induces a bounded composition operator*

$$\varphi^* : L^{1,p}(\mathbb{D}) \to L^{1,q(p,s)}(\Omega)$$

for any $p \in (2; +\infty)$ and $q(p,s) = ps/(p+s-2)$.

Remark. Bertilsson proved in 1999 that the Brennan conjecture holds when $4/3 < p < 3.422$ but the full result remains open.)

In this part we study spectral properties of divergence form elliptic operators $-\mathrm{div}[A(z)\nabla f(z)]$ with Neumann boundary conditions in a large class of planar (non)convex domains $\Omega \subset \mathbb{C}$. The suggested method is based on the composition operators on Sobolev spaces and its applications to the Poincaré-Sobolev inequalities.

Classical Neumann problem:

$$L_A = -\mathrm{div}[A(z)\nabla f(z)], \quad z=(x,y)\in\Omega, \quad \left.\frac{\partial f}{\partial n}\right|_{\partial\Omega} = 0,$$

in a smooth domain $\Omega \subset \mathbb{C}$ with $A \in M^{2\times 2}(\Omega)$. We denote here by $M^{2\times 2}(\Omega)$, the class of all 2×2 symmetric matrix functions $A(z) = \{a_{kl}(z)\}$, $\det A = 1$, with measurable entries satisfying the uniform ellipticity condition

$$\frac{1}{K}|\xi|^2 \leq \langle A(z)\xi, \xi\rangle \leq K|\xi|^2 \text{ a.e. in } \Omega,$$

for every $\xi \in \mathbb{C}$, where $1 \leq K < \infty$.

For non smooth domain we consider the weak formulation of the Neumann eigenvalue problem:

$$\iint\limits_{\Omega} \left\langle A(z)\nabla f(z), \nabla\overline{g(z)} \right\rangle dxdy = \mu \iint\limits_{\Omega} f(z)\overline{g(z)}\, dxdy, \ \ \forall g \in W^{1,2}_A(\Omega).$$

Here $L^{1,2}_A(\Omega)$ defined as the space of all locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ with the finite seminorm:

$$\|f \mid L^{1,2}_A(\Omega)\| = \left(\iint\limits_{\Omega} \langle A(z)\nabla f(z), \nabla f(z)\rangle \ dxdy \right)^{\frac{1}{2}}.$$

Sobolev space $W^{1,2}_A(\Omega)$ is a normed space endowed with the following norm:

$$\|f \mid W^{1,2}_A(\Omega)\| = \|f \mid L^2(\Omega)\| + \|f \mid L^{1,2}_A(\Omega)\|.$$

By the Min–Max Principle for the first non-trivial Neumann eigenvalue $\mu_1(A,\Omega)$ we have $\mu_1(A,\Omega)^{-\frac{1}{2}} = B_{2,2}(A,\Omega)$ where $B_{2,2}(A,\Omega)$ is the best constant in the following Poincaré inequality

$$\inf_{c\in\mathbb{R}} \|f - c \mid L^2(\Omega)\| \leq B_{2,2}(A,\Omega)\|f \mid L^{1,2}_A(\Omega)\|, \quad f \in W^{1,2}_A(\Omega).$$

The classical upper estimate for the first non-trivial Neumann eigenvalue of the Laplace operator

$$\mu_1(I,\Omega) := \mu_1(\Omega) \leq \mu_1(\Omega^*) = \frac{j'^2_{1,1}}{R^2_*}$$

was proved by Szegö for simply connected planar domains via a conformal mappings technique ("the method of conformal normalization"). In this inequality $j'_{1,1}$ denotes the first positive zero of the derivative of the Bessel function J_1 and Ω^* is a disc of the same area as Ω with R_* as its radius.

In convex domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the classical lower estimates of the Neumann eigenvalues of the Laplace operator states that

$$\mu_1(\Omega) \geq \frac{\pi^2}{d(\Omega)^2},$$

where $d(\Omega)$ is a diameter of a convex domain Ω. Similar estimates for the non-linear p-Laplace operator, $p \neq 2$, were obtained much later.

The composition operators reduce the spectral problem in a simply connected domain $\Omega \subset \mathbb{C}$ to a weighted spectral problem for the Laplace operator in the unit disc $\mathbb{D} \subset \mathbb{C}$. By the chain rule for the function $f(z) = g \circ \varphi(z)$, we have

$$-\mathrm{div}[A(z)\nabla f(z)] = -\mathrm{div}[A(z)\nabla g(\varphi(z))] = -\left|J(w, \varphi^{-1})\right|^{-1} \Delta g(w),$$

where $J(w, \varphi^{-1})$ denotes the Jacobian of the inverse mapping $\varphi^{-1} : \mathbb{D} \to \Omega$.

Example. Consider $-\mathrm{div}[A(z)\nabla f(z)]$ with the matrix

$$A(z)=\begin{pmatrix} \frac{a+b}{a-b} & 0 \\ 0 & \frac{a-b}{a+b} \end{pmatrix},\ a>b\geq 0,$$

defined in the interior of ellipse Ω_e with semi-axes $a+b$ and $a-b$. By ou main theorem

$$\mu_1(A,\Omega_e)\geq \frac{(j'_{1,1})^2}{a^2-b^2},$$

what is better then the lower classical estimate :

$$\mu_1(A,\Omega_e)\geq \frac{\pi^2}{4(a+b)^2}\frac{a-b}{a+b}.$$

For thin ellipses, i.e $a+b$ fixed and $(a-b)$ tends to zero an asymptotic of our estimate is ∞ when for the classical one it is 0.

The application of the composition operators theory to spectral problems of the A-divergent form elliptic operators is based on reducing of a positive quadratic form

$$ds^2 = a_{11}(x,y)dx^2 + 2a_{12}(x,y)dxdy + a_{22}(x,y)dy^2$$

defined in a planar domain Ω, by means of a quasiconformal change of variables, to the canonical form

$$ds^2 = \Lambda(du^2 + dv^2),\ \Lambda \neq 0,\ \text{a.e. in } \Omega',$$

given that $a_{11}a_{22} - a_{12}^2 \geq \kappa_0 > 0$, $a_{11} > 0$, almost everywhere in Ω.

Let a quasiconformal mapping satisfies $\varphi(z)$, to the Beltrami equation:

$$\varphi_{\overline{z}}(z) = \mu(z)\varphi_z(z), \text{ a.e. in } \Omega,$$

with the complex dilatation $\mu(z)$ given by the following way

$$\mu(z) = \frac{a_{22}(z) - a_{11}(z) - 2ia_{12}(z)}{\det(I + A(z))}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We call this quasiconformal mapping (with the complex dilatation μ) as an A-quasiconformal mapping.

The uniform ellipticity condition can be reformulated as

$$|\mu(z)| \leq \frac{K-1}{K+1}, \text{ a.e. in } \Omega.$$

Conversely, the following representation of the matrix A :

$$A(z) = \begin{pmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2Im\mu}{1-|\mu|^2} \\ \frac{-2Im\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}, \text{ a.e. in } \Omega.$$

Theorem. Let Ω, Ω' be domains in $\mathbb{C}$. Then a homeomorphism $\varphi : \Omega \to \Omega'$ is an A-quasiconformal mapping if and only if φ induces, by the composition rule $\varphi^*(f) = f \circ \varphi$, an isometry of Sobolev spaces $L^{1,2}_A(\Omega)$ and $L^{1,2}(\Omega')$:

$$\|\varphi^*(f) \mid L^{1,2}_A(\Omega)\| = \|f \mid L^{1,2}(\Omega')\|$$

for any $f \in L^{1,2}(\Omega')$.

This theorem generalizes the well known property of conformal mappings generate an isometry of uniform Sobolev spaces $L^1_2(\Omega)$ and $L^1_2(\Omega')$ and refines (in the case $n = 2$) the functional characterization of quasiconformal mappings in the terms of isomorphisms of uniform Sobolev spaces.
An exact formulation will be later.

If $\varphi : \Omega \to \mathbb{D}$ is an A-quasiconformal mapping, then the inverse mapping $\psi = \varphi^{-1} : \mathbb{D} \to \Omega$ is A^{-1}-quasiconformal.
Denote by $B_{r,2}(\mathbb{D})$, $1 < r < \infty$, the best constant in the (non-weighted) Poincaré-Sobolev inequality in the unit disc $\mathbb{D}$.
Exact calculations of $B_{r,2}(\mathbb{D})$, $r \neq 2$, is an open problem and we use the upper estimate:

$$B_{r,2}(\mathbb{D}) \leq \left(2^{-1}\pi\right)^{\frac{2-r}{2r}} (r+2)^{\frac{r+2}{2r}} .$$

Denote by $h(z) = |J(z,\varphi)|$ the quasihyperbolic weight defined by an A-quasiconformal mapping $\varphi : \Omega \to \mathbb{D}$.

Theorem. Let a matrix A satisfies the uniform ellipticity condition and Ω be a simply connected planar domain. Then for any function $f \in W^{1,2}_A(\Omega)$ the weighted Poincaré-Sobolev inequality

$$\inf_{c\in\mathbb{R}} \left(\iint\limits_{\Omega} |f(z) - c|^r h(z) dxdy \right)^{\frac{1}{r}}$$

$$\leq B_{r,2}(h, A, \Omega) \left(\iint\limits_{\Omega} \langle A(z)\nabla f(z), \nabla f(z)\rangle \, dxdy \right)^{\frac{1}{2}}$$

holds for any $r \geq 1$ with the constant $B_{r,2}(h, A, \Omega) = B_{r,2}(\mathbb{D})$.

A simply connected domain $\Omega \subset \mathbb{C}$ is called an A-quasiconformal β-regular domain, $\beta > 1$, if

$$\iint\limits_{\mathbb{D}} |J(w, \varphi^{-1})|^{\beta} \, dudv < \infty,$$

where $\varphi : \Omega \to \mathbb{D}$ is an A-quasiconformal mapping.

Theorem. Let a matrix A satisfies the uniform ellipticity condition and Ω be an A-quasiconformal β-regular domain. Then:

1. the embedding operator

$$i_\Omega : W^{1,2}_A(\Omega) \hookrightarrow L^s(\Omega)$$

   is compact for any $s \geq 1$;

2. for any function $f \in W^{1,2}_A(\Omega)$ and for any $s \geq 1$, the Poincaré–Sobolev inequality

$$\inf_{c \in \mathbb{R}} \|f - c \mid L^s(\Omega)\| \leq B_{s,2}(A,\Omega) \|f \mid L^{1,2}_A(\Omega)\|$$

Here

$$B_{s,2}(A,\Omega) \leq B_{\frac{\beta s}{\beta-1},2}(\mathbb{D}) \| J_{\varphi^{-1}} \mid L^{\beta}(\mathbb{D}) \|^{\frac{1}{s}},$$

where $J_{\varphi^{-1}}$ is a Jacobian of the quasiconformal mapping $\varphi^{-1} : \mathbb{D} \to \Omega$.

Theorem Let A be a matrix satisfies the uniform ellipticity condition and Ω be an A-quasiconformal β-regular domain. Then the spectrum of the Neumann divergence form elliptic operator L_A in Ω is discrete, and can be written in the form of a non-decreasing sequence:

$$0 = \mu_0(A,\Omega) < \mu_1(A,\Omega) \leq \mu_2(A,\Omega) \leq \ldots \leq \mu_n(A,\Omega) \leq \ldots,$$

and

$$\frac{1}{\mu_1(A,\Omega)} \leq B_{\frac{2\beta}{\beta-1},2}(\mathbb{D}) \| J_{\varphi^{-1}} \mid L^{\beta}(\mathbb{D})\|$$

$$\leq \frac{4}{\sqrt[\beta]{\pi}} \left(\frac{2\beta-1}{\beta-1}\right)^{\frac{2\beta-1}{\beta}} \left\| J_{\varphi^{-1}} \mid L^{\beta}(\mathbb{D})\right\|.$$

The homeomorphism

$$\varphi(z) = \frac{z^{\frac{3}{2}}}{\sqrt{2} \cdot \overline{z}^{\frac{1}{2}}} - 1,\ \varphi(0) = -1, \quad z = x + iy,$$

is an A-quasiconformal and maps the interior of the "rose petal"

$$\Omega_p := \left\{ (\rho, \theta) \in \mathbb{R}^2 : \rho = 2\sqrt{2}\cos(2\theta), \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \right\}$$

onto the unit disc $\mathbb{D}$.
Here $|J(w, \varphi^{-1})| = |J(z, \varphi)|^{-1} = 1$. Then by the main theorem we have

$$\frac{1}{\mu_1(A, \Omega_p)} \leq \frac{1}{(j'_{1,1})^2} \operatorname{esssup}_{|w|<1} |J(w, \varphi^{-1})| = \frac{1}{(j'_{1,1})^2}.$$

Spectral estimates in quasidiscs (images of the unit disc under quasiconformal mapping $\psi : R^2 \to R^2$).
For K-quasidisc we have the following estimate

$$\mu_1(A, \Omega) \geq \frac{M(K)}{|\Omega|}.$$

Here the constant $M(K)$ depends on the coefficient quasiconformality K only.

$$M(K) := \frac{\pi}{K^2} \exp\left\{-\frac{K^2\pi^2(2+\pi^2)^2}{2\log 3}\right\} \inf_{1<\beta<\beta^*}$$

$$\left\{\left(\frac{2\beta-1}{\beta-1}\right)^{-\frac{2\beta-1}{\beta}} C_\beta^{-2}\right\},$$

$$C_\beta = \frac{10^6}{[(2\beta-1)(1-\nu(\beta))]^{1/2\beta}},$$

where $\beta^* = \min\left(\frac{K}{K-1}, \widetilde{\beta}\right)$, and $\widetilde{\beta}$ is the unique solution of the equation

$$\nu(\beta) := 10^{8\beta}\frac{2\beta-2}{2\beta-1}(24\pi^2K^2)^{2\beta} = 1.$$

Two A-quasiconformal β-regular domains Ω_1, Ω_2 represent an *A-quasiconformal β-regular pair* if there exists an A-quasiconformal mappings $\psi : \Omega_1 \to \Omega_2$ such that

$$\iint\limits_{\Omega_1} |J(z,\psi)|^{\beta} dxdy < \infty \ \& \ \iint\limits_{\Omega_2} |J(w,\psi^{-1})|^{\beta} dudv < \infty.$$

Note, that two A-quasiconformal β-regular domains $\Omega_1 = \varphi_1^{-1}(\mathbb{D})$ and $\Omega_2 = \varphi_2^{-1}(\mathbb{D})$ represent an A-quasiconformal β-regular pair if and only if

$$\Phi_{\beta}(\varphi_1,\varphi_2) = \left(\iint\limits_{\mathbb{D}} \max\left\{ \frac{|J(z,\varphi_1^{-1}|^{\beta}}{|J(z,\varphi_2^{-1}|^{\beta-1}}, \frac{|J(z,\varphi_2^{-1}|^{\beta}}{|J(z,\varphi_1^{-1}|^{\beta-1}} \right\} \right)^{\frac{1}{2\beta}} < \infty.$$

As an example, two Ahlfors type domains also represent an A-quasiconformal β-regular pair.

We proved that if $\Omega_1 = \varphi_1^{-1}(\mathbb{D})$ and $\Omega_2 = \varphi_2^{-1}(\mathbb{D})$ represent an A-quasiconformal β-regular pair, then for any $n \in \mathbb{N}$:

$$|\mu_n[A,\Omega_1] - \mu_n[A,\Omega_2]| \leq c_n B^2_{\frac{4\beta}{\beta-1},2}(\mathbb{D},h)\Phi_\beta(\varphi_1,\varphi_2)\cdot \| J^{\frac{1}{2}}_{\varphi_1^{-1}} - J^{\frac{1}{2}}_{\varphi_2^{-1}} \mid L^2(\mathbb{D})\|,$$

where $c_n = \max\left\{\mu_n^2[A,\Omega_1], \mu_n^2[A,\Omega_2]\right\}$, $J_{\varphi_k^{-1}}$ are Jacobians of A^{-1}-quasiconformal mappings $\varphi_k^{-1} : \mathbb{D} \to \Omega_k$, $k = 1,2$.

Remark. The constant $B_{\frac{4\beta}{\beta-1},2}(\mathbb{D},h)$ is the best constant for corresponding weighted Poincaré-Sobolev inequalities in the unit disc $\mathbb{D}$.

Nonlinear Elasticity (G. Ukhlov 2019-2020)

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $\varphi : \Omega \to \mathbb{R}^n$ be a mapping. In fundamental papers (1976-1981) J. M. Ball demonstrated that solutions of typical boundary valued nonlinear elasticity problems are minimizers of complicated energy integrals of deformations

$$I(\varphi, \Omega) := \int_{\Omega} W(x, \varphi(x), D\varphi(x))\, dx,$$

where $\varphi : \Omega \to \mathbb{R}^n$ is a continuous weak differentiable mapping (deformation).

In the spatial case ($n = 3$) J. M. Ball demonstrated that appropriate classes of minimizers (so-called Ball's classes) are

$$A^{+}_{q,r}(\Omega) = \left\{\varphi \in L^1_q(\Omega) : \operatorname{adj} D\varphi \in L_r(\Omega),\ J(x,\varphi) > 0 \text{ a. e. } x \in \Omega\right\},$$

where $q > n-1$ and $r \geq q/(q-1)$. He studied invertibility of deformations for $q > n$ in the case of incompressible bodies ($J(x,\varphi) \equiv 1$) and its weak regularity. Namely φ^{-1} belongs to the class $L^1_1(\widetilde{\Omega})$ where $\widetilde{\Omega} = \varphi(\Omega)$.

In the case $q = n$ continuity of Ball's mapping follows from G. and Vodopjanov work (1979) (using the property $J(x\varphi) > 0$).
In the work of V.Sverak (1988) was proved invertibility of Ball's classes for for $(q \geq n-1)$. This case $(n-1 < q \leq n)$ is more complicated. V.Sverak also proved that Ball's mapping are homeomorphisms outside of so-called "singular" sets S of (n-q)-Hausdorff measure zero and the inverse mapping is continuous outside a set of (n-1)-dimensional Hausdorff measure zero. The Sverak's proof contains two gaps. He did not put attention that his change of variables formula is correct only if the mapping φ posses the Luzin N-property. He proved only that for any discontinuity point $x \in S$ limiting values of φ has (n-1)-dimensional Hausdorff measure zero.

Hajlasz and Koskela (2003) proved that q-dimensional capacity of the singular set S is zero and $(n-1)$-dimensional Hausdorff measure of its "image" is also zero. It means that after the deformation the body Ω has not cracks. In the present work we use the corrected results of Sverak and we prove that topological mappings (homeomorphisms) of Ball's classes which possess the Luzin N-property are absolutely continuous with respect to capacity (which considered as an outer measure associated with the Sobolev spaces). This result is based on an interpretation of Ball's classes in terms of composition operators on Sobolev spaces $L^1_p(\Omega)$ (and/or (p,q)-quasiconformal mappings, that induce such operators). The Ball's classes correspond to the case (∞, p). It lead us to a generalization of Ball's classes for arbitrary values of $1 \leq p, q \leq \infty$.

We show that a topological mapping of Sobolev classes $\varphi : \Omega \to \widetilde{\Omega}$, $J(x, \varphi) > 0$ for almost all $x \in \Omega$, belongs to $A^{+}_{q,r}(\Omega)$ if and only if φ possesses the Luzin N-property and for any function $f \in L^1_\infty(\widetilde{\Omega})$ (a bounded gradient of a stress tensor) the composition $\varphi^*(f) = f \circ \varphi \in L^1_q(\Omega)$ and

$$\|f \mid L^1_1(\widetilde{\Omega})\| \leq \| \operatorname{adj} D\varphi | L_r(\Omega)\| \cdot \|\varphi^*(f) \mid L^1_q(\Omega)\|,$$

where $r = q/(q-1)$.

This inequality states that the second Ball's condition $\operatorname{adj} D\varphi \in L_r(\Omega)$, $r = q/(q-1)$ is equivalent to the boundedness of the composition operator

$$\left(\varphi^{-1}\right)^{*} : L^1_q(\Omega) \to L^1_1(\widetilde{\Omega}).$$

generated by the inverse topological mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$.
Remark. The mapping φ is a homeomorphism from $\Omega \setminus S$ into $\widetilde{\Omega}$.

This property leads to corresponding capacity estimates:

$$\operatorname{cap}_1(\varphi(E);\widetilde{\Omega}) \leq \| \operatorname{adj} D\varphi | L_r(\Omega)\| \cdot \operatorname{cap}_q^{1/q}(E;\Omega)$$

for any q-capacity measurable set $E \subset \Omega$.
It means that topological mappings of Ball's classes are absolutely continuous with respect to capacity and an image of a cavitation of q-capacity zero ("singular" set) in a body Ω has 1-capacity zero (also "singular" set) in $\widetilde{\Omega}$ and can not lead to the body breaking upon these deformations.

A condenser in a closure $\bar{\Omega}$ of a domain $\Omega \subset \mathbb{R}^n$ is the pair (F_0, F_1) of connected disjoint closed relatively to $\bar{\Omega}$ sets $F_0, F_1 \subset \bar{\Omega}$. A continuous function $f \in L^1_p(\Omega)$ is called an admissible function for the condenser (F_0, F_1), if the set $F_i \cap \bar{\Omega}$, is contained in some connected component of the set $\bar{\operatorname{Int}}\{x | f(x) = i\}$, $i = 0, 1$. We call as the p-capacity of the condenser (F_0, F_1) relatively to domain Ω the following quantity:

$$\operatorname{cap}_p(F_0, F_1; \Omega) = \inf \|f | L^1_p(\Omega)\|^p.$$

Here the greatest lower bond is taken over all functions admissible for the condenser $(F_0, F_1) \subset \Omega$.

So, the cavitation can be characterized in capacitary terms. As a consequence we obtain characterization of cavitations in the terms of the Hausdorff measure: for every set $E \subset \Omega$ such that $\operatorname{cap}_q(E;\Omega) = 0$, $1 \leq p \leq n$, the Hausdorff measure $H^{n-1}(\varphi(E)) = 0$. This is an explanation why $q > n-1$ in the definition of Ball's classes.

Remark. Mappings of the Ball's classes are mappings of finite distortion due to the property $J(x,\varphi) > 0$ for almost all $x \in \Omega$.

The functional definition of Ball's classes.

We can define the Ball classes $A^{+}_{q,r}(\Omega)$ as class of Sobolev mappings $\varphi : \Omega \to \widetilde{\Omega}$, $J(x,\varphi) > 0$ for almost all $x \in \Omega$, such that

$$\|f \mid L^1_1(\widetilde{\Omega})\| \leq \| \operatorname{adj} D\varphi | L_r(\Omega)\| \cdot \|\varphi^*(f) \mid L^1_q(\Omega)\|,$$

for any $f \in L^1_\infty(\widetilde{\Omega})$, $q > n-1$, $r \geq q/(q-1)$.
For the case $n-1 < q < n$ it is necessary to add Luzin N-property into this definition.

Let $\varphi : \Omega \to \widetilde{\Omega}$ be a topological mapping of Sobolev class. We proved in 2010 that $\varphi \in L^1_q(\Omega)$ if and only if the corresponding composition operator ("change of variables")

$$\varphi^* : L^1_\infty(\widetilde{\Omega}) \to L^1_q(\Omega)$$

is bounded. It means that the first Ball's condition have a reinterpretation in terms of composition operators. Recall that L^1_∞ is a space of all Lipschitz functions.

We proved that the second Balls condition
$\operatorname{adj} D\varphi \in L_r(\Omega)\},\ q \geq n-1,\ r \geq \frac{q}{q-1}$ is equivalent to the boundedness of the composition operator

$$\left(\varphi^{-1}\right)^{*} : L^1_q(\Omega) \to L^1_1(\widetilde{\Omega}).$$

Therefore we have a "linearization" of Ball's classes in terms of linear composition operators.

Let us consider the case $n = 3$ and $q = n - 1 = 2$. Then the second condition is equivalent to the following condition

$$\left(\int\limits_{\widetilde{\Omega}} \frac{|D\varphi^{-1}(y)|^2}{|J(y, \varphi^{-1})|} \, dy \right)^{\frac{1}{2}} < \infty,$$

and it means that a ratio of deformations of line elements to deformations of (co-dimension 1) surface elements must be integrable, that seems to be natural from the point of view of mechanics of elastic bodies.

Remark. Let $\lambda_3 \leq \lambda_2 \leq \lambda_1$ are singular values of $D\varphi^{-1}(y)$ then $\frac{|D\varphi^{-1}(y)|^2}{|J(y,\varphi^{-1})|} = \frac{\lambda_1}{\lambda_2 \lambda_3}$

We propose the following generalization of Ball's classes as classes of mappings with finite integral (p,q)-variation of the non-linear elastic potential energy:

$$K^{I}_{p,q}(\Omega)^{-1}\|f \mid L^1_q(\widetilde{\Omega})\| \leq \|\varphi^*(f) \mid L^1_p(\Omega)\| \leq K_{p,q}(\Omega)\|f \mid L^1_\infty(\widetilde{\Omega})\|, \ 1 \leq q \leq p \leq \infty, \quad (1)$$

where $K_{p,q}$ and $K^{I}_{p,q}$ are the integral distortion and the integral inner distortion of corresponding weak (p,q)-quasiconformal mappings.

Inner distortion.

Let $\varphi : \Omega \to \widetilde{\Omega}$ be a mapping of finite distortion of the class $W^1_{1,\mathrm{loc}}(\Omega)$. We define the inner q-distortion of φ at a point x as

$$K_q^I(x,\varphi) = \begin{cases} \frac{|J(x,\varphi)|^{\frac{1}{q}}}{l(D\varphi(x))}, & J(x,\varphi) \neq 0, \\ 0, & J(x,\varphi) = 0. \end{cases}$$

Here $l(D\varphi(x))$ is a minimal singular value of $D\varphi(x)$ i.e. $\min\limits_{h=1} |D\varphi(x) \cdot h|$

The global integral version of the inner distortion we call an inner (p,q)-distortion, $1 \leq q \leq p \leq \infty$:

$$K^I_{p,q}(\Omega) = \|K^I_q(\varphi) \mid L_\kappa(\Omega)\|, \; 1/\kappa = 1/q - 1/p, \; (\kappa = \infty, \text{ if } p = q).$$

Remark. Let $\varphi \in L^1_p(\Omega)$, $q = 1$ and $J(x,\varphi) > 0$. Using $(D\varphi(x))^{-1} = J^{-1}(x,\varphi) \operatorname{adj} D\varphi(x)$ and $\min\limits_{h=1} |D\varphi(x) \cdot h| = \left(\max\limits_{h=1} |(D\varphi(x))^{-1} \cdot h| \right)^{-1}$ we obtain

$$\left(K^I_{p,1}(\Omega)\right)^{\frac{p}{p-1}} = \int\limits_\Omega \left(\frac{|J(x,\varphi)|}{l(D\varphi(x))} \right)^{\frac{p}{p-1}} dx = \int\limits_\Omega |\operatorname{adj} D\varphi(x)|^{\frac{p}{p-1}} \, dx < \infty$$

Next theorem explains the generalization of Ball's classes.

Theorem. Let $\varphi : \Omega \to \widetilde{\Omega}$ be a homeomorphism of finite distortion and $\varphi \in L^1_p(\Omega)$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates a bounded composition operator

$$\left(\varphi^{-1}\right)^* : L^1_p(\Omega) \to L^1_q(\widetilde{\Omega}),\ 1 \leq q < p < \infty,$$

if and only if $\varphi^{-1} \in L^1_q(\widetilde{\Omega})$, possesses the Luzin N^{-1}-property and

$$K^I_{p,q}(\Omega) = \left(\int\limits_{\Omega} K^I_q(x,\varphi)^{\frac{pq}{p-q}}\ dx\right)^{\frac{p-q}{pq}} < \infty.$$

Theorem. Let a homeomorphism of finite distortion $\varphi : \Omega \to \widetilde{\Omega}$ belong to $L^1_p(\Omega)$, $1 < p < \infty$, possess the Luzin N-property and such that

$$K^I_{p,q}(\Omega) = \|K^I_q(\varphi) \mid L_\kappa(\Omega)\| < \infty,\ 1 < q < p < \infty,$$

where $1/\kappa = 1/q - 1/p$. Then for every condenser $(F_0, F_1) \subset \Omega$ the inequality

$$\operatorname{cap}_q^{\frac{1}{q}}(\varphi(F_0), \varphi(F_1); \widetilde{\Omega}) \leq K^I_{p,q}(\Omega) \operatorname{cap}_p^{\frac{1}{p}}(F_0, F_1; \Omega)$$

Theorem. Let a homeomorphism of finite distortion $\varphi : \Omega \to \widetilde{\Omega}$ belong to $L^1_p(\Omega)$, $1 < p < \infty$, possess the Luzin N-property and such that

$$K^I_{p,q}(\Omega) = \|K^I_q(\varphi) \mid L_\kappa(\Omega)\| < \infty, \ 1 < q < p < \infty,$$

where $1/\kappa = 1/q - 1/p$. Then for every set $E \subset \Omega$ such that $\operatorname{cap}_p(E;\Omega) = 0$, $1 \leq p \leq n$, the Hausdorff measure $H^\alpha(\varphi(E)) = 0$ for any $\alpha > n - q$.

Some consequences for mappings φ of Ball's classes $A^{+}_{q,r}(\Omega)$. For the direct mapping φ the singular set has the q-capacity zero and has the α-Hausdorff measure zero for any $\alpha > n - q$.
Inverse mapping induce the composition operator

$$\left(\varphi^{-1}\right)^{*} : L^1_q(\Omega) \to L^1_1(\widetilde{\Omega}).$$

Hence $\widetilde{\Omega} \setminus \varphi(\Omega)$ has 1-capacity zero and has α-Hausdorff measure zero for any $\alpha > n - 1$ and can not produce cracks.

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THANKS